COMBINATORICA
Akadémiai Kiadó – Springer-Verlag

COMBINATORICA **14** (3) (1994) 321–334

RESTRICTED GREEDY CLIQUE DECOMPOSITIONS AND GREEDY CLIQUE DECOMPOSITIONS OF K_4-FREE GRAPHS

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Received January 25, 1992
Revised April 1, 1993

A greedy clique decomposition of a graph is obtained by removing maximal cliques from a graph one by one until the graph is empty. It has recently been shown that any greedy clique decomposition of a graph of order n has at most $\frac{n^2}{4}$ cliques. In this paper, we extend this result by showing that for any positive integer p, $3 \le p$ any clique decomposition of a graph of order n obtained by removing maximal cliques of order at least p one by one until none remain, in which case the remaining edges are removed one by one, has at most $t_{p-1}(n)$ cliques. Here $t_{p-1}(n)$ is the number of edges in the Turán graph of order n, which has no complete subgraphs of order p.

In connection with greedy clique decompositions, P. Winkler conjectured that for any greedy clique decomposition $\mathscr{C}$ of a graph G of order n the sum over the number of vertices in each clique of $\mathscr{C}$ is at most $\frac{n^2}{2}$. We prove this conjecture for K_4-free graphs and show that in the case of equality for $\mathscr{C}$ and G there are only two possibilities:

(i) $G \simeq K_{n/2,n/2}$

(ii) G is complete 3-partite, where each part has $n/3$ vertices.

We show that in either case $\mathscr{C}$ is completely determined.

1. Introduction

For a graph G we denote its vertex set by $V(G)$, its edge set by $E(G)$ and denote the cardinalities of $V(G)$ and $E(G)$ by $n(G)$ and $m(G)$, respectively. By a *clique* of G we shall mean a complete subgraph of G, and by a *clique decomposition* of G we shall mean a collection of cliques which partition $E(G)$. An *ordered clique decomposition* of G is a pair $(\mathscr{C} \prec)$ where $\mathscr{C}$ is a clique decomposition of G and $\prec$ is a total ordering defined on $\mathscr{C}$. An ordered clique decomposition $(\mathscr{C},\prec)$ where $\mathscr{C}$ is attained by removing maximal cliques (i.e. their edges) one by one until the graph is empty, and $\prec$ coincides with the order in which maximal cliques are removed, is called a *greedy clique decomposition*. For a greedy clique decomposition we shall always assume that maximal cliques which are single edges are picked last. Extending this definition further, for a positive integer p, $3 \le p$ we define a *greedy p-decomposition*. A greedy p-decomposition is an ordered clique decomposition $(\mathscr{C},\prec)$, where $\mathscr{C}$ is attained by removing maximal cliques of order at least p one by one until none remain, in which case the remaining edges are removed one by one. The order $\prec$ coincides with the order in which cliques were removed. One

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AMS subject classification code (1991): 05 C 35 immediately sees that a greedy clique decomposition of a graph G is also a greedy 3-decomposition.

A graph is said to be *r-partite* if its vertices can be divided into r parts, where no two vertices in a given part are adjacent. A graph is *complete r-partite* if its vertices can be divided into r parts, where two vertices are adjacent if and only if they belong to different a parts. We denote by K_r the complete graph on r vertices, and we denote the complete r-partite graph with parts of sizes $n_1, n_2, \ldots, n_r$ by $K_{n_1,n_2,\ldots,n_r}$. We say a graph is *K_r-free* if it does not contain K_r as a subgraph.

For $2 \leq r \leq n$ we let $T_r(n)$ denote the Turán graph on n vertices, where

$$T_r(n) \simeq K_{\lfloor \frac{n}{r} \rfloor, \lfloor \frac{n+1}{r} \rfloor, \ldots, \lfloor \frac{n+r-1}{r} \rfloor}.$$

We let $t_r(n) = m(T_r(n))$, and remark that $T_r(n)$ is the unique r-partite graph of order n which has a maximum number of edges. It was shown by Turán [8] that $T_r(n)$ is the unique graph of order n with at least $t_r(n)$ edges which is K_{r+1}-free.

A classic result of Erdős, Goodman, and Pósa [4] states that any graph of order n has a clique decomposition with at most $\frac{n^2}{4}$ cliques. In fact, their proof indicated that this can be achieved with at most $\frac{n^2}{4}$ edges and triangles. Bollobás [2] subsequently strengthened this result by showing that for any positive integer $r \geq 3$, any graph of order n can be partioned into at most $t_{r-1}(n)$ K_r's and edges.

It was recently shown by the author [7] that any greedy clique decomposition of a graph of order n has at most $\frac{n^2}{4}$ cliques. This result settled a conjecture of Winkler [9]. In this paper, we extend this result by showing for a positive integer p, $3 \leq p$ any greedy p-decomposition of a graph of order n has at most $t_{p-1}(n)$ cliques. In particular, it follows from this that any greedy p-decomposition of a K_{p+1}-free graph has at most $t_{p-1}(n)$ K_p's and edges. This is interesting in light of Bollobás' result in that it says that for K_{p+1}-free graphs even if we pick as many K_p's as possible into the clique decomposition by choosing at random, Bollobás' result still applies; that is, such a clique decomposition still has at most $t_{p-1}(n)$ cliques. In [6], Győri and Tuza improved Bollobás' result by showing that for $p \geq 4$ and any graph G of order n, there exists a clique decomposition $\mathscr{C}$ of G consisting solely of K_p's and edges where

$$\sum_{X \in \mathscr{C}} n(X) \leq 2t_{p-1}(n).$$

In the case when $p = 3$, the above result is not true as it was shown in [6] that every clique decomposition $\mathscr{C}$ of K_{6m+4} into triangles and edges has $\sum_{X \in \mathscr{C}} n(X) \geq 2t_2(6m+4)+1$. However, it was conjectured in [6] that for any graph of order n, such a clique decomposition $\mathscr{C}$ into edges and triangles exists for which $\sum_{X \in \mathscr{C}} n(X) \leq 2t_2(n) + o(n^2)$.

In conjunction with the above results, we have a conjecture of Winkler [9]:

Conjecture 1.1. *For any greedy clique decomposition $(\mathscr{C}, \prec)$ of a graph of order n, $\sum_{X \in \mathscr{C}} n(X) \leq \frac{n^2}{2}$.*

It was shown by Chung [3] and independently by Győri and Kostochka [5] that for any graph of order n there exists a clique decomposition $\mathscr{C}$ such that

$$\sum_{X\in\mathscr{C}} n(X) \le \frac{n^2}{2}.$$

In the second part of this paper, we show that Conjecture 1.1 is true for K_4-free graphs with equality holding in the above for a greedy clique decomposition $(\mathscr{C},\prec)$ of a graph G if and only if

(i) $G\simeq K_{n/2,n/2}$ or

(ii) $G\simeq K_{n/3,n/3,n/3}$.

Moreover, in (i) and (ii) $\mathscr{C}$ is completely determined. In (i), $\mathscr{C}$ is simply all the edges of G. In (ii), G can be expressed as a complete 3-partite graph with parts $V_i\cup W_i$, $i=1,2,3$, where $V_i\cap W_i=\emptyset$, $|V_i|=|W_i|=\frac{n}{6}$, $i=1,2,3$, and the triangles of $\mathscr{C}$ cover exactly the edges between V_i and V_j and W_i and W_j, for $i<j$.

As an interesting corollary of the above, it follows that for any K_4-free graph G of order n where $m(G)=\frac{n^2}{4}+k$, $0\le k\le\frac{n^2}{12}$ one can pick at least $\frac{2k}{3}$ edge-disjoint triangles simply by choosing at random.

We will first introduce some notation. For a graph G and $A\subseteq E(G)$ (respectively, $A\subseteq V(G)$) we say H is a subgraph *induced* by A if H is the subgraph of G where $E(H)=A$ and $V(H)$ is the union of all endvertices of edges of A (respectively, $V(H)=A$ and $E(H)$ is the set of edges having both its endvertices in A). For $S\subseteq V(G)$, we denote the set of neighbours of S in G by $N_G(S)$ which is the set of vertices of $V(G)-S$ which are adjacent to at least one vertex in S. We say a set of vertices S is *independent* if no two vertices in S are adjacent.

For a clique decomposition $\mathscr{C}$ and $i=1,2,3,\dots$ we let $\mathscr{C}^i$ denote the set of cliques of $\mathscr{C}$ of order i. For each vertex v we let $\mathscr{C}_v$ denote the set of cliques of $\mathscr{C}$ containing v and for each edge e we let $\mathscr{C}_e$ be the set of cliques of $\mathscr{C}$ containing an endvertex of e. Finally, for $i=2,3,4,\dots$, and for all $v\in V(G)$ and $e\in E(G)$ we let $\mathscr{C}_v^i=\mathscr{C}^i\cap\mathscr{C}_v$ and $\mathscr{C}_e^i=\mathscr{C}^i\cap\mathscr{C}_e$.

All graphs in this paper will be assumed to be simple (loopless, no multiple edges). We will often denote an edge by its endvertices, as for example if an edge e has endvertices x and y we will often write xy instead of e. A triangle with vertices x, y, and z will often just be denoted by xyz.

2. Restricted greedy clique decompositions

In [7], we showed that for any greedy clique decomposition $(\mathscr{C},\prec)$ of a graph G of order n that $\sum_{C\in\mathscr{C}} n(C)\le\frac{n^2}{4}$. The proof is by induction on n: we first find a clique $C_e\in\mathscr{C}^2$ and remove its vertices, and we remove the edges of the cliques in $\mathscr{C}_e$. This results in a graph H and a greedy clique decomposition $\mathscr{C}'=\mathscr{C}-\mathscr{C}_e$ of H. Since by hypothesis $|\mathscr{C}'|\le\frac{(n-2)^2}{4}$, we need only show that $|\mathscr{C}_e|\le n-1$. This we do by viewing the cliques of $\mathscr{C}_e$ as subsets of vertices, and by showing that there exists a transversal which excludes a vertex. We do this by assigning to C_e an arbitrary endpoint of e, and to every clique $C\in\mathscr{C}_e$ we assign either a vertex of $V(C)$ which occurs in no other clique of $\mathscr{C}_e$, or if no such one exists, we assign to C any vertex which occurs in another clique $D\in\mathscr{C}_e-\{C_e,C\}$ for which $D\prec C$. The maximality of C when it was chosen guarantees that at least one such vertex exists, and it is also seen that this assignment defines a transversal on $\mathscr{C}_e$ which excludes an endpoint of e. In this section, we refine the above proof and use it for greedy p-decompositions.

Let G be a graph and let $(\mathscr{C},\prec)$ be an ordered clique decomposition of G. For each edge e of G we define a function ψ from $\mathscr{C}_e$ to the set of all subsets of $V(G)$, including the empty set. Let $e\in E(G)$ and suppose C_e is the clique of $\mathscr{C}$ covering e. We set $\psi(C_e)=V(C_e)$, and for all $X\in\mathscr{C}_e-\{C_e\}$ we let $\psi(X)$ be the set of vertices of $V(X)$ which either belong to no other cliques of $\mathscr{C}_e-\{X\}$, or belong to some clique $D\in\mathscr{C}_e-\{X,C_e\}$ for which $D\prec X$. Here we observe that each vertex of $V(G)-V(C_e)$ belongs to at most two cliques of $\mathscr{C}_e$, and hence at most one such clique D exists.

Lemma 2.1. *For all $e\in E(G)$*

$$|\mathscr{C}_e|=\left|\bigcup_{X\in\mathscr{C}_e}V(X)\right|-\sum_{X\in\mathscr{C}_e}(|\psi(X)|-1).$$

Proof. The sets $\psi(X)$, $X\in\mathscr{C}_e$ are seen to partition the set $\bigcup_{X\in\mathscr{C}_e}V(X)$ and thus

$$\sum_{X\in\mathscr{C}_e}(|\psi(X)|-1)+|\mathscr{C}_e|=\sum_{X\in\mathscr{C}_e}|\psi(X)|=\left|\bigcup_{X\in\mathscr{C}_e}V(X)\right|.$$ ∎

Let G, $\mathscr{C}$, e, and C_e be as above. We can use Lemma 2.1 to prove the following.

Proposition 2.2. *If $\psi(X)\neq\emptyset$ for all $X\in\mathscr{C}_e$, then $|\mathscr{C}_e|\le\left|\bigcup_{X\in\mathscr{C}_e}V(X)\right|-n(C_e)+1$. Also, if $(\mathscr{C},\prec)$ is a greedy p-decomposition for G and $C_e=e$, then $\psi(X)\neq\emptyset$ for all $X\in\mathscr{C}_e-\mathscr{C}_e^2$ and*

$$|\mathscr{C}_e|\le\left|\bigcup_{X\in\mathscr{C}_e}V(X)\right|-\sum_{X\in\mathscr{C}_e^2}(|\psi(X)|-1).$$

Proof. The first part follows directly from Lemma 2.1 and the fact that $\psi(C_e)=V(C_e)$.

In the second part, let $e=uv$. Suppose $X\in\mathscr{C}_e-\mathscr{C}_e^2$. If $V(X)-\bigcup_{Y\in\mathscr{C}_e-\{X\}}V(Y)\neq\emptyset$, then by definition of ψ, $\psi(X)\neq\emptyset$. Therefore suppose that $V(X)\subseteq\bigcup_{Y\in\mathscr{C}_e-\{X\}}V(Y)$. Then $V(X)\cup\{u,v\}$ induces a clique X' of order $n(X')=n(X)+1$. Let D be the first clique chosen into $\mathscr{C}$ which covers some edges of X'. Since when each clique of $\mathscr{C}-\mathscr{C}^2$ was chosen it was maximal, D cannot be properly contained in X', and thus $D \neq X$ and $D \neq e$, It then follows that $D \prec X$ and D meets X' at exactly one edge uy or vy, depending on whether $X \in \mathscr{C}_v$ or $X \in \mathscr{C}_u$, respectively. By definition of ψ, we see that $y \in \psi(X)$ and thus $\psi(X) \neq \emptyset$. We conclude thus that for all $X \in \mathscr{C}_e - \mathscr{C}_e^2$, $\psi(X) \neq \emptyset$ and now the second part follows from Lemma 2.1. ∎

Let $3 \leq p$ and let $(\mathscr{C}, \prec)$ be a greedy p-decomposition of a graph G. From Proposition 2.2 we see that for $e \in \mathscr{C}^2$, if $X \in \mathscr{C}_e$ and $\psi(X) = \emptyset$, then $X \in \mathscr{C}_e^2$. For each $e \in \mathscr{C}^2$ we define α_e to be the number of cliques $X \in \mathscr{C}_e^2$ for which $\psi(X) = \emptyset$. It is simple observation that α_e equals the number of pairs of edges in $\mathscr{C}_e^2$ which share a common vertex, not being an endvertex of e. As a quick observation, we have the following corollary which is a direct consequence of the proof of Proposition 2.2.

Corollary 2.3. *If $(\mathscr{C}, \prec)$ is a greedy clique decomposition of a graph G, then $\alpha_e = 0$ for all $e \in \mathscr{C}^2$. Moreover, if $e \in \mathscr{C}^2$ and $|\mathscr{C}_e| + 1 = n(G)$, then $|\psi(X)| = 1$ for all $C \in \mathscr{C}_e - \{e\}$ and $\bigcup_{X \in \mathscr{C}_e} V(X) = V(G)$.* ∎

We shall now prove the first of the main results.

Theorem 2.4. *Let $3 \leq p$ and let G be a graph of order n. If $(\mathscr{C}, \prec)$ is a greedy p-decomposition of G, then $|\mathscr{C}| \leq t_{p-1}(n)$.*

Proof. By induction on n. If $n \leq p$, the theorem is seen to be true. We therefore assume that $n > p$ and the theorem is true for any graph of order less than n.

Let $(\mathscr{C}, \prec)$ be a greedy p-decomposition of a graph G of order n. Suppose $e = uv \in \mathscr{C}^2$ and define a graph H by $H = G - \{u, v\} - \bigcup_{X \in \mathscr{C}_e} E(X)$. Let $(\mathscr{C}', \prec')$ be the ordered clique decomposition of H where $\mathscr{C}' = \mathscr{C} - \mathscr{C}_e$ and $\prec'$ is the same as $\prec$ restricted to $\mathscr{C}'$. Then $(\mathscr{C}', \prec')$ is a greedy p-decomposition for H and thus, by the inductive assumption, $|\mathscr{C}'| \geq t_{p-1}(n-2)$. If $|\mathscr{C}_e| \leq t_{p-1}(n) - t_{p-1}(n-2)$, then

$$|\mathscr{C}| = |\mathscr{C}'| + |\mathscr{C}_e| \leq t_{p-1}(n-2) + t_{p-1}(n) - t_{p-1}(n-2) = t_{p-1}(n).$$

Thus we may assume for all $e \in \mathscr{C}^2$ that

$$|\mathscr{C}_e| \geq 1 + t_{p-1}(n) - t_{p-1}(n-2). \tag{2.1}$$

We have from Lemma 2.1 that for $e \in \mathscr{C}^2$

$$1 + \sum_{X \in \mathscr{C}_e - \mathscr{C}_e^2} (|\psi(X)| - 1) - \alpha_e + |\mathscr{C}_e| = \left| \bigcup_{X \in \mathscr{C}_e} V(X) \right|. \tag{2.2}$$

By Proposition 2.2, for $e \in \mathscr{C}^2$, $\psi(X) \neq \emptyset$ for all $X \in \mathscr{C}_e - \mathscr{C}^2$. Since $\left| \bigcup_{X \in \mathscr{C}_e} V(X) \right| \leq n$ we now obtain from (2.2) that for all $e \in \mathscr{C}^2$

$$\alpha_e \geq |\mathscr{C}_e| - n + 1. \tag{2.3}$$

Now (2.1) and (2.3) together imply that for all $e \in \mathscr{C}^2$

$$\alpha_e \geq 2 + t_{p-1}(n) - t_{p-1}(n-2) - n. \tag{2.4}$$

Let $k > 0$ and $0 \leq \ell < p-1$ be integers such that $n = k(p-1) + \ell$. Then we have that

$$t_{p-1}(n) - t_{p-1}(n-2) = 2(n-k) - 3 + \max\{2-\ell, 0\}. \tag{2.5}$$

From (2.4) and (2.5) we obtain for all $e \in \mathscr{C}^2$ that

$$\alpha_e \geq n - 2k - 1 + \max\{2-\ell, 0\}. \tag{2.6}$$

If $\mathscr{C}^i = \emptyset$ for all $i \geq p$, then G must be K_p-free and hence by Turán's Theorem [8], $|\mathscr{C}| = m(G) \leq t_{p-1}(n)$. Thus we may assume that there exists p^*, $p \leq p^*$ such that $\mathscr{C}^{p^*} \neq \emptyset$. Let $L \in \mathscr{C}^{p^*}$ and let $u \in V(L)$. If $\mathscr{C}_u^2 = \emptyset$, then $|\mathscr{C}_u| \leq \frac{n-1}{p-1}$. In this case, let $H = G - u - \bigcup_{X \in \mathscr{C}_u} E(X)$ and let $(\mathscr{C}', \prec')$ be the ordered clique decomposition of H where $\mathscr{C}' = \mathscr{C} - \mathscr{C}_u$, and $\prec'$ is the same as $\prec$ restricted to $\mathscr{C}'$. Then $(\mathscr{C}', \prec')$ is a greedy p-decomposition for H, and thus by the inductive assumption $|\mathscr{C}'| \leq t_{p-1}(n-1)$. It now follows that $|\mathscr{C}| = |\mathscr{C}'| + |\mathscr{C}_u| \leq t_{p-1}(n-1) + \frac{n-1}{p-1} \leq t_{p-1}(n)$. We may thus assume $\mathscr{C}_u^2 \neq \emptyset$, and we let $f = uv \in \mathscr{C}_u^2$.

Let J be a maximal clique containing f in the subgraph induced by the edges of $\mathscr{C}^2$. For all $x \in V(G) - V(J)$ let β_x be the number of edges $xy \in \mathscr{C}^2$ where $y \in V(J)$. Counting in two ways we obtain

$$\sum_{e \in E(J)} \alpha_e = m(J) \cdot (n(J) - 2) + \sum_{x \in V(G) - V(J)} \binom{\beta_x}{2} \tag{2.7}$$

We note that by the maximality of J, $\beta_x \leq n(J) - 1$ for all $x \in V(G) - V(J)$. For each $x \in V(J) - u$ let γ_x be the number of edges $xy \in \mathscr{C}^2$ where $y \in V(L) - \{u\}$. Then

$$\sum_{x \in V(L) - u} \beta_x = \sum_{x \in V(J) - u} \gamma_x.$$

Suppose for some $x \in V(J) - u$, $\gamma_x = n(L) - 1$. Then $V(L) \cup \{x\}$ induces a clique of order $n(L) + 1$ which properly contains L. But now $xy \in \mathscr{C}^2$ for all $y \in V(L)$, and thus L could not have been maximal when it was chosen; a contradiction. Thus for all $x \in V(J) - \{u\}$, $\gamma_x \leq n(L) - 2$, and therefore

$$\sum_{x \in V(L) - u} \beta_x = \sum_{x \in v(J) - u} \gamma_x \leq (n(J) - 1)(n(L) - 2). \tag{2.8}$$

It is now straightforward to prove that

$$\sum_{x \in V(L) - u} \binom{\beta_x}{2} \leq (n(L) - 2)\binom{n(J) - 1}{2}. \tag{2.9}$$

By (2.6), (2.7) and (2.9) we have

$$m(J)(n-2k-1+\max\{2-\ell,0\}) \le \sum_{e\in E(J)} \alpha_e$$

$$\le m(J)(n(J)-2)+(n-n(J)-1)\cdot\binom{n(J)-1}{2}. \tag{2.10}$$

Dividing the left and right side of (2.10) by $m(J)=\binom{n(J)}{2}$ we have

$$n-2k-1+\max\{2-\ell,0\} \le n(J)-2+(n-n(J)-1)\frac{(n(J)-2)}{n(J)}$$

$$= n-1-\frac{2(n-1)}{n(J)}.$$

Thus

$$n(J) \ge \frac{2(n-1)}{2k-\max\{2-\ell,0\}}. \tag{2.11}$$

Since $n=k(p-1)+\ell$, (2.11) becomes

$$n(J) \ge \frac{2k(p-1)+2(\ell-1)}{2k-\max\{2-\ell,0\}} > p-1.$$

Thus $n(J)\ge p$, but this gives a contradiction since by the nature of $\mathscr{C}$ the subgraph induced by $\mathscr{C}^2$ must be K_p-free. It must therefore be the case that $|\mathscr{C}|\le t_{p-1}(n)$, and this completes the induction. ∎

As an immediate consequence of Theorem 2.4 we have the following corollary:

Corollary 2.5. *For $p\ge 3$ and any K_{p+1}-free graph of order n, a clique decomposition formed by choosing at random as many K_p's as possible and then choosing the remaining edges gives at most $t_{p-1}(n)$ edges and K_p's.* ∎

As mentioned before, Bollobás [2] proved that for $p\ge 3$ and any graph G of order n there exists a clique decomposition of G consisting of K_p's and edges which has at most $t_{p-1}(n)$ cliques. Corollary 2.5 states that for K_{p+1}-free graphs Bollobás' result still holds even if we pick our clique decomposition by first picking at random as many edge disjoint K_p's as possible. The question naturally arises, how many edge disjoint K_p's can one be assured of getting simply by choosing at random? In connection with this question, it follows from a result of Bollobás [2] that if G is a graph of order n, and

$$m(G)=y\left(\frac{n}{p-1}\right)^2\binom{p-1}{2}+(1-y)\left(\frac{n}{p}\right)^2\binom{p}{2}$$

where $0\le y\le 1$, then G has at least $(1-y)(\frac{n}{p})^p$ K_p's as subgraphs. That is, if

$$m(G)=\left(\frac{n}{p-1}\right)^2\binom{p-1}{2}+k$$

where

$$0 \le k < \binom{n}{p}\binom{p}{2} - \left(\frac{n}{p-1}\right)^2 \binom{p-1}{2}$$

then Bollobás' result implies G has at least $\frac{n^{p-2}2(p-1)}{p^{p-1}}k$ K_p's as subgraphs. Assuming G is K_{p+1}-free, any K_p is easily seen to intersect (in edges) at most

$$\binom{n-p}{p-2} + \binom{n-p}{p-3} + \ldots + \binom{n-p}{1} \sim \frac{n^{p-2}}{(p-2)!} + o(n^{p-2})$$

other K_p's. Thus Bollobás' result implies that we can pick at least

$$\sim \frac{n^{p-2}2(p-1)}{p^{p-1}}k \cdot \frac{(p-2)!}{n^{p-2}(1+o(1))} = \frac{2(p-1)!k}{p^{p-1}(1+o(1))}$$

edge disjoint K_p's simply by choosing at random.

If G is K_{p+1}-free, Corollary 2.5 implies any greedy p-decomposition $\mathscr{C}$ of G has at most $t_{p-1}(n)$ cliques. Thus

$$\begin{aligned}|\mathscr{C}| &= |\mathscr{C}^2| + |\mathscr{C}^p| = m(G) - \binom{p}{2}|\mathscr{C}^p| + |\mathscr{C}^p| \\ &= \left(\frac{n}{p-1}\right)^2 \binom{p-1}{2} + k - \binom{p}{2}|\mathscr{C}^p| + |\mathscr{C}^p| \le t_{p-1}(n).\end{aligned}$$

So

$$|\mathscr{C}^p| \ge \frac{\left(\frac{n}{p-1}\right)^2 \cdot \binom{p-1}{2} + k - t_{p-1}(n)}{\binom{p}{2} - 1}.$$

Setting $n = k(p-1) + \ell$, $0 \le \ell < p-1$ we find that $\left(\frac{n}{p-1}\right)^2 \binom{p-1}{2} - t_{p-1}(n) = \frac{\ell}{2}\left(1 - \frac{\ell}{p-1}\right) \ge 0$. Thus $|\mathscr{C}^p| \ge \frac{k}{\binom{p}{2}-1}$ by Corollary 2.5. We note further that

$$\frac{k}{\binom{p}{2} - 1} > \frac{2(p-1)!}{p^{p-1}}k, \quad p \ge 3.$$

The lower bound implied by Corollary 2.5 is seen to be better than that implied by Bollobás Theorem.

3. Greedy clique decomposition of K_4-free graphs

In this section, we prove Conjecture 1.1 for K_4-free graphs.

Let $\mathscr{C}$ be a clique decompostion of a graph G. By counting in two ways we obtain the following relationship:

$$\sum_{X \in \mathscr{C}} n(X) = \sum_{v \in V(G)} |\mathscr{C}_v|. \tag{3.1}$$

We define a subgraph $G_{\mathscr{C}}$ of G by letting $V(G_{\mathscr{C}})=V(G)$ and letting uv be an edge of $G_{\mathscr{C}}$ if and only if $|\mathscr{C}_u|+|\mathscr{C}_v|\leq n(G)$. We call a vertex v *positive* (respectively, *non-positive*) with respect to $\mathscr{C}$ if $|\mathscr{C}_v|>\frac{n(G)}{2}$ (respectively $|\mathscr{C}_v|\leq\frac{n(G)}{2}$).

Lemma 3.1. *Let $\mathscr{C}$ be a clique decomposition of a graph G of order n. If there exists a matching in $G_{\mathscr{C}}$ which covers all positive vertices, then*

$$\sum_{X\in\mathscr{C}} n(X)\leq\frac{n^2}{2}.$$

Proof. Suppose we have a matching in $G_{\mathscr{C}}$ which covers vertices $W\subseteq V(G)$, and W contains all positive vertices. Then for any edge $uv\in E(G_{\mathscr{C}})$ in the matching, $|\mathscr{C}_u|+|\mathscr{C}_v|\leq n$, and thus by (3.1)

$$\begin{aligned}\sum_{X\in\mathscr{C}} n(x) &= \sum_{x\in V(G)}|\mathscr{C}_x|\\ &=\sum_{x\in W}|\mathscr{C}_x|+\sum_{x\in V(G)-W}|\mathscr{C}_x|\\ &\leq\frac{|W|}{2}\cdot n+(n-|W|)\cdot\frac{n}{2}=\frac{n^2}{2}.\end{aligned}$$

∎

As a corollary we have:

Corollary 3.2. *If $(\mathscr{C},\prec)$ is a greedy decomposition of a graph G of order n, and $\mathscr{C}^2$ contains a perfect matching of G, then*

$$\sum_{X\in\mathscr{C}} n(X)\leq\frac{n^2}{2}.$$

Proof. From Corollary 2.3, $\alpha_e=0$ for all $e\in\mathscr{C}^2$ and by Proposition 2.2 for $e\in\mathscr{C}^2$, $\psi(X)\neq\emptyset$ for all $X\in\mathscr{C}^2_e$. Thus by Proposition 2.2, for all $e\in\mathscr{C}^2$

$$|\mathscr{C}_e|\leq\left|\bigcup_{X\in\mathscr{C}_e}V(X)\right|-1\leq n-1.$$

Thus if $e=uv\in\mathscr{C}^2$, then $|\mathscr{C}_u|+|\mathscr{C}_v|\leq n$ and $e\in E(G_{\mathscr{C}})$. It now follows that a perfect matching of G consisting of edges from $\mathscr{C}^2$ is also a matching in $G_{\mathscr{C}}$ and the corollary now follows from Lemma 3.1. ∎

For the remainder of this section, G will denote a K_4-free graph of G (which has only edges and triangles). For an edge $e=uv$ we define a function $\pi_e:\mathscr{C}_e\to\mathscr{C}_e$ (or just simply π when e is implicit) in the following manner. Suppose $X\in\mathscr{C}_e$. If no cliques of $\mathscr{C}_e-\{X\}$ have vertices in $V(X)-\{u,v\}$ then we define $\pi_e(X)=X$. If there exists $D\in\mathscr{C}_e-\{X\}$ such that $V(X)\cap V(D)-\{u,v\}\neq\emptyset$, then there can be at most one such clique D since G is K_4-free, and we define $\pi_e(X)=D$ in this case. We see that π_e is well-defined, and bijective.

We shall use the following key lemma:

Lemma 3.3. *Let $e=uv$ be an edge where $|\mathscr{C}_e|\geq n-1$. If C_e is the clique of $\mathscr{C}$ covering e, then there are at least $|\mathscr{C}_u^3|+n(C_e)-3$ cliques $X\in\mathscr{C}_v^2$ for which $\pi(X)\neq X$.*

Proof. Similar to (2.2) we have

$$n(C_e)-1+\sum_{X\in\mathscr{C}_e^3-\{C_e\}}(|\psi(X)|-1)-\alpha_e+|\mathscr{C}_e|=\left|\bigcup_{X\in\mathscr{C}_e}V(X)\right|. \tag{3.2}$$

Since $\left|\bigcup_{X\in\mathscr{C}_e}V(X)\right|\leq n$ and $|\mathscr{C}_e|\leq n-1$, (3.2) implies

$$\alpha_e\geq\sum_{X\in\mathscr{C}_e^3-\{C_e\}}(|\psi(X)|-1)+n(C_e)-2. \tag{3.3}$$

Let

$$B_1=\{X\in\mathscr{C}_u^3-\{C_3\}:\pi(X)=X\},$$
$$B_2=\{X\in\mathscr{C}_u^3-\{C_3\}:\pi(X)\in\mathscr{C}_v^2\}$$

and

$$B_3=\{X\in\mathscr{C}_u^3-\{C_3\}:\pi(X)\in\mathscr{C}_v^3\}.$$

By (3.3.),

$$\begin{aligned}\alpha_e\geq n(C_e)-2&+\sum_{X\in B_1}(|\psi(X)|-1)+\sum_{X\in B_2}(|\psi(X)|-1)\\ &+\sum_{X\in B_3}(|\psi(X)|+|\psi(\pi(X))|-2).\end{aligned} \tag{3.4}$$

For each $X\in B_1$, $|\psi(X)|=2$ and for each $X\in B_2$, $|\psi(X)|=1$. For each $X\in B_3$, $|\psi(X)|+|\psi(\pi(X))|-2=1$. Thus by (3.4) we have

$$\alpha_e\geq|B_1|+|B_3|+n(C_e)-2.$$

The number of cliques $X\in\mathscr{C}_v^2$ for which $\pi(X)\neq X$ is seen to be $\alpha_e+|B_2|$. Thus, by the above this number is seen to be at least $|B_1|+|B_2|+|B_3|+n(C_e)-2=|\mathscr{C}_u^3|+n(C_e)-3$. ∎

As a consequence of Lemma 3.3 we have the following:

Proposition 3.4. *If $e\in\mathscr{C}^2$ and $|\mathscr{C}_e|=n-1$ then $\pi(X)\in\mathscr{C}_e^2$ for all $X\in\mathscr{C}_e^3$, and $\bigcup_{X\in\mathscr{C}_e}V(X)=V(G)$.*

Furthermore, if $T=uvw\in\mathscr{C}^3$, where $|\mathscr{C}_u|+|\mathscr{C}_w|\geq n$ and $|\mathscr{C}_v|+|\mathscr{C}_w|\geq n$, then $|\mathscr{C}_w^2|\geq|\mathscr{C}_u^3|+|\mathscr{C}_v^3|$.

Proof. For the first part, suppose $e\in\mathscr{C}^2$ and $|\mathscr{C}_e|=n-1$. Then by Corollary 2.3, $|\psi(X)|=1$ for all $X\in\mathscr{C}_e-\{e\}$ and $\bigcup_{X\in\mathscr{C}_e}V(X)=V(G)$. Thus for $X\in\mathscr{C}_e^3$, $|\psi(X)|=1$

and therefore $\pi(X)\neq X$. If $X\in\mathscr{C}_e^3$ and $\pi(X)\in\mathscr{C}_e^3$, then $|\psi(X)|+|\psi(\pi(X))|-2=1$, and hence either $|\psi(X)|>1$ or $|\psi(\pi(X))|>1$. Thus for all $X\in\mathscr{C}_e^3$, $\pi(X)\in\mathscr{C}_e^2$.

For the second part, suppose $T=uvw\in\mathscr{C}^3$ where $|\mathscr{C}_u|+|\mathscr{C}_w|\geq n$ and $|\mathscr{C}_v|+|\mathscr{C}_w|\geq n$. Let S_{uw} be the set of vertices belonging to both an edge of $\mathscr{C}_w^2$ and a clique of $\mathscr{C}_u-\{T\}$. For $f=uw$, the number of cliques $X\in\mathscr{C}_w^2$ for which $\pi_f(X)\neq X$ equals $|S_{uw}|$, and thus by Lemma 3.3, $|S_{uw}|\geq|\mathscr{C}_u^3|+n(T)-3=|\mathscr{C}_u^3|$. Defining S_{vw} in a similar way we see that $|S_{vw}|\geq|\mathscr{C}_v^3|$. Since G is K_4-free, $S_{uw}\cap S_{vw}=\emptyset$, and therefore

$$|\mathscr{C}_w^2|\geq|S_{uw}\cup S_{vw}|=|S_{vw}|+|S_{vw}|\geq|\mathscr{C}_u^3|+|\mathscr{C}_v^3|.$$ ∎

We now give the second main result of this paper.

Theorem 3.5. *Let G be a K_4-free graph of order n and let $(\mathscr{C},\prec)$ be a greedy decomposition of G. Then $\sum_{X\in\mathscr{C}} n(X)\leq\frac{n^2}{2}$ and equality holds if and only if either*

(i) *$G\simeq K_{\frac{n}{2},\frac{n}{2}}$ and $\mathscr{C}$ consists of the edges of G or*

(ii) *G can be expressed as a complete 3-partite graph having parts $V_i\cup W_i$, $i=1,2,3$ where $V_i\cap W_i=\emptyset$, $|V_i|=|W_i|=\frac{n}{6}$, and the triangles of $\mathscr{C}$ cover exactly the edges of*

$$\bigcup_{i<j}\{xy:x\in V_i,\,y\in V_j\}\cup\{xy:x\in W_i,\,y\in W_j\}.$$

Proof. We shall first show that there exists a matching in $G_{\mathscr{C}}$ which covers all positive vertices of G. It will then follow by Lemma 3.1 that $\sum_{X\in\mathscr{C}} n(X)\leq\frac{n^2}{2}$.

Let S be a nonempty subset of positive vertices. For simplicity we will write $N(S)$ in place of $N_{G_{\mathscr{C}}}(S)$, the set of neighbours of S in $G_{\mathscr{C}}$. We shall now show that $|N(S)|\geq|S|+1$. Suppose the contrary holds; that is, $|N(S)|\leq|S|$. For each $x\in V(G)$ let $\mathscr{C}'_x$ be the set of cliques of $\mathscr{C}_x$ which contain no vertices of $N(S)$, and for each edge e let $\mathscr{C}'_e$ be the set of cliques of $\mathscr{C}_e$ which contain no vertices of $N(S)$. Suppose $x\in S$. Then by Proposition 2.2, for each $e\in\mathscr{C}_x^2$.

$$|\mathscr{C}_e|\leq\left|\bigcup_{X\in\mathscr{C}_e}|V(X)|\right|+\sum_{X\in\mathscr{C}_e^2}(|\psi(X)|-1)-1.$$

Moreover, by Corollary 2.3, $\alpha_e=0$ and thus $|\psi(X)|=1$ for all $X\in\mathscr{C}_e^2$. The above now implies $|\mathscr{C}_e|\leq n-1$, and thus $e\in E(G_{\mathscr{C}})$ for all $e\in\mathscr{C}_x^2$. We now deduce that $\mathscr{C}'_x\subseteq\mathscr{C}_x^3$ for all $x\in S$ and furthermore

$$|\mathscr{C}_x^3|\geq|\mathscr{C}'_x|\geq|\mathscr{C}_x|-|N(S)|.$$

Let $u\in S$ be an arbitrary vertex in S. Then by the above

$$2|\mathscr{C}'_u|\geq 2|\mathscr{C}_u|-2|N(S)|\geq n+1-|N(S)|-|S|.$$

Since there are at most $\frac{n-|N(S)|-|S|}{2}$ triangles of $\mathscr{C}'_u$ not containing vertices in $S-u$, the above implies that there exists at least one triangle $T \in \mathscr{C}'_u$ which contains vertices of $S-u$. Let $T = uvw$ where $v \in S$, and $w \in V(G) - N(S)$. Since G is K_4-free, the number of cliques incident with T which contain vertices of $N(S)$ is at most $2|N(S)|$. Thus

$$|\mathscr{C}_{uv} - \mathscr{C}'_{uv}| + |\mathscr{C}_w - \mathscr{C}'_w| \leq 2|N(S)|,$$

$$|\mathscr{C}_{uv} - \mathscr{C}'_{uv}| \leq 2|N(S)| - |\mathscr{C}_w - \mathscr{C}'_w|$$

and

$$|\mathscr{C}'_{uv}| \geq |\mathscr{C}_{uv}| - 2|N(S)| + |\mathscr{C}_w - \mathscr{C}'_w|. \tag{3.5}$$

Since $\mathscr{C}'_{uv} \subset \mathscr{C}^3_{uv}$ and $|\mathscr{C}_{uv}| \geq n$ (because u and v are positive) we obtain from (3.5) that

$$\begin{aligned} |\mathscr{C}^3_u| + |\mathscr{C}^3_v| = |\mathscr{C}^3_{uv}| + 1 &\geq |\mathscr{C}'_{uv}| + 1 \\ &\geq n + 1 - 2|N(S)| + |\mathscr{C}_w - \mathscr{C}'_w|. \end{aligned} \tag{3.6}$$

By Proposition 3.4, $|\mathscr{C}^2_w| \geq |\mathscr{C}^3_u| + |\mathscr{C}^3_v|$ and thus it follows from (3.6) that

$$|\mathscr{C}^2_w| \geq n + 1 - 2|N(S)| + |\mathscr{C}_w - \mathscr{C}'_w|.$$

That is,

$$\begin{aligned} |\mathscr{C}^2_w \cap \mathscr{C}'_w| \geq |\mathscr{C}^2_w| - |\mathscr{C}_w - \mathscr{C}'_w| &\geq n + 1 - 2|N(S)| \\ &\geq n + 1 - |N(S)| - |S|. \end{aligned} \tag{3.7}$$

Suppose $wy \in \mathscr{C}^2_w \cap \mathscr{C}'_w$. Then $y \notin N(S)$ (by definition of $\mathscr{C}'_w$). Since $wy \in \mathscr{C}^2_w$, we have that $wy \in E(G_{\mathscr{C}})$, and $|\mathscr{C}_w| + |\mathscr{C}_y| \leq n$. Thus w and y cannot be both positive, and therefore $y \notin S$. Thus $y \in V(G) - N(S) - S$, and we now see that $|\mathscr{C}^2_w \cap \mathscr{C}'_w| \leq n - |N(S)| - |S|$ which contradicts (3.7). Thus we have shown using proof by contradiction that $|N(S)| \geq |S| + 1$.

Since $|N(S)| > |S|$ for every nonempty set S of positive vertices, it follows from applying Hall's Theorem [1] that there exists a matching in $G_{\mathscr{C}}$ which covers all positive vertices. This completes the proof of the first part.

Suppose now that $\sum_{X \in \mathscr{C}} n(X) = \frac{n^2}{2}$, and let u be an arbitrary nonpositive vertex. Since $|N(S)| \geq |S|+1$ for any nonempty subset S of positive vertices, it follows that $|N(S) - u| \geq |S|$ for all subsets S. Therefore, by applying Hall's Theorem again, we deduce that there exists a matching in $G_{\mathscr{C}} - u$ which covers all positive vertices. Let W be the set of vertices covered by one such matching. Then, as shown in Lemma 3.1

$$\begin{aligned} \frac{n^2}{2} = \sum_{X \in \mathscr{C}} n(X) &= \sum_{v \in W} |\mathscr{C}_v| + \sum_{v \in V(G) - W} |\mathscr{C}_v| \\ &\leq \frac{|W|}{2} \cdot n + (n - |W|) \cdot \frac{n}{2} = \frac{n^2}{2}. \end{aligned} \tag{3.8}$$

Thus equality holds everywhere in (3.8) and therefore $|\mathscr{C}_v|=\frac{n}{2}$ for all $v\in V(G)-W$. Since $u\in V(G)-W$, $|\mathscr{C}_u|=\frac{n}{2}$, and since u was an arbitrarily chosen non-positive vertex, it follows that $|\mathscr{C}_u|=\frac{n}{2}$ for all non-positive vertices u. Now, since $\frac{n^2}{2}=\sum_{X\in\mathscr{C}} n(X)=\sum_{v\in V(G)} |\mathscr{C}_v|$, we must have that $|\mathscr{C}_v|=\frac{n}{2}$ for all v.

We shall now show that G and $\mathscr{C}$ satisfy either (i) or (ii) of Theorem 3.5.

Since for all $e\in\mathscr{C}^2$ we have $|\mathscr{C}_e|=n-1$, Proposition 3.4 implies that for all $e\in\mathscr{C}^2$ and for all $X\in\mathscr{C}_e^3$

$$\pi(X)\in\mathscr{C}_e^2-\{e\} \tag{3.9}$$

and

$$V(G)=\bigcup_{X\in\mathscr{C}_e} V(X). \tag{3.10}$$

Now, if $e=uv\in\mathscr{C}^2$, and $T=vyw\in\mathscr{C}_v^3$, then (3.9) implies $\pi_e(T)\in\mathscr{C}_u^2-\{uv\}$ and hence either $uy\in\mathscr{C}^2$ or $uw\in\mathscr{C}^2$. That is, if $e=uv\in\mathscr{C}^2$ and $T=vyw\in\mathscr{C}^3$, then either

$$uy\in\mathscr{C}^2 \text{ or } uw\in\mathscr{C}^2. \tag{3.11}$$

If $\mathscr{C}$ contains no triangles, then G is K_3-free, $\frac{n^2}{4}=|\mathscr{C}|=m(G)$, and therefore (i) holds.

Suppose $\mathscr{C}$ contains a triangle $T=uvw$. Let $S=\{x: xu\in\mathscr{C}^2$, or $xv\in\mathscr{C}^2$, or $xw\in\mathscr{C}^2\}$. Let $x\in S$, and suppose $xu\in\mathscr{C}^2$. Then by (3.11) either $xv\in\mathscr{C}^2$ or $xw\in\mathscr{C}^2$. More generally, each vertex of S is joined to T by exactly two edges in $\mathscr{C}^2$. Let S_{uv}, S_{uw}, S_{vw} be a partition of S, S_{xy} being the set of vertices of S joined to x and y. Using Proposition 3.4, we have three inequalities: $|\mathscr{C}_w^2|\ge|\mathscr{C}_u^3|+|\mathscr{C}_v^3|$, $|\mathscr{C}_u^2|\ge|\mathscr{C}_w^3|+|\mathscr{C}_v^3|$, and $|\mathscr{C}_v^2|\ge|\mathscr{C}_u^3|+|\mathscr{C}_w^3|$. By adding these together we obtain

$$|\mathscr{C}_u^2|+|\mathscr{C}_v^2|+|C_w^2|\ge 2(|\mathscr{C}_u^3|+|\mathscr{C}_v^3|+|C_w^3|).$$

Adding $2(|\mathscr{C}_u^2|+|\mathscr{C}_v^2|+|\mathscr{C}_w^2|)$ to both sides, we obtain

$$3(|\mathscr{C}_u^2|+|\mathscr{C}_v^2|+|\mathscr{C}_w^2|)\ge 2(|\mathscr{C}_u|+|\mathscr{C}_v|+|\mathscr{C}_w|)=3n.$$

Then $3\cdot 2|S|\ge 3n$, and $|S|\ge\frac{n}{2}$.

Suppose $a\in S_{uw}$ and $b\in S_{uv}$. Then $f=aw\in\mathscr{C}^2$. Now $wb\notin E(G)$, for otherwise b, u, v, and w induce a K_4. By (3.10), $b\in\bigcup_{X\in\mathscr{C}_f} V(X)$ and since $wb\neq E(G)$, it follows that $ab\in E(G)$. Now $ab\notin\mathscr{C}^2$ for otherwise ab, ua, ub would be edges of $\mathscr{C}^2$ inducing a triangle. Thus ab is covered by a triangle in $\mathscr{C}$, say $T'=abc$. Since $vb\in\mathscr{C}^2$ and $va\notin E(G)$, (3.11) implies $vc\in\mathscr{C}^2$, and therefore $c\in S$. Since G is K_4-free, S_{uv}, S_{uw}, and S_{vw} are independent, and thus $c\in S_{vw}$. We see in general that S_{uv}, S_{uw}, and S_{vw} are three parts inducing a complete 3-partite graph, the edges between the sets being partitioned by triangles of $\mathscr{C}$. Furthermore, we note that all parts have the same size $\frac{|S|}{3}$.

Corresponding to sets S, S_{uw}, S_{uv}, and S_{vw} for T, we have sets S', S_{ab}, S_{bc}, and S_{ac} for T', which have the corresponding properties. Similar to that for S, we have $|S'| \geq \frac{n}{2}$, and now since clearly $S \cap S' = \emptyset$, we deduce $|S| = |S'| = \frac{n}{2}$, $S \cup S' = V(G)$, and $|S_{xy}| = \frac{n}{6}$ for all x, y.

Consider vertices $x \in S_{ab} - \{u\}$ and $y \in S_{uw} - \{a\}$. Since $xa \in \mathscr{C}^2$, (3.10) implies $ay \in E(G)$ or $xy \in E(G)$. The former is impossible since S_{uw} is independent, so $xy \in E(G)$, and in fact $xy \in \mathscr{C}^2$ since it clearly cannot be covered by a triangle of $\mathscr{C}$. The same conclusion holds if $y \in S_{uv} - \{b\}$. Suppose $y \in S_{vw} - \{c\}$. If $xy \in E(G)$, then clearly $xy \in \mathscr{C}^2$ and with same reasoning as above we deduce that $uc \in \mathscr{C}^2$; a contradiction. Thus $xy \notin E(G)$. It is now seen that in general $S_{ab} \cup S_{vw}$, $S_{ac} \cup S_{uv}$, and $S_{bc} \cup S_{uw}$ form the parts of a complete 3-partite graph, the triangles of $\mathscr{C}$ covering exactly those edges between the sets S_{ab}, S_{ac}, S_{bc} and those between S_{vw}, S_{uv}, S_{uw}. Thus G and $\mathscr{C}$ are as in (ii). ∎

In closing, we broaden Conjecture 1.1 which in view of Theorem 2.4 seems natural:

Conjecture 3.6. *For any greedy p-decomposition $(\mathscr{C}, \prec)$ of a graph of order n,*

$$\sum_{X \in \mathscr{C}} n(X) \leq 2t_{p-1}(n).$$

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